

Differential Equations with Fuzzy Parameters via Differential Inclusions

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We give a definition of solutions of ordinary differential equations in \mathbb{R}^n containing parameters which are described by changing in time fuzzy sets. They are defined as fuzzy subsets of the space of absolutely continuous functions. We introduce a hypograph metric in the space of fuzzy sets and prove a theorem on continuous dependence of fuzzy solutions on parameters and initial conditions with respect to that metric. © 2001 Academic Press

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1. INTRODUCTION

A certain number of papers have appeared where attempts have been made to investigate ordinary differential equations with uncertainty about some of their components (initial value, parameters) described in terms of fuzzy sets. Different approaches have been applied. In [16, 17, 21, 25] (see also [14]), the authors constructed a new differential equation in the space of fuzzy sets using a notion of the derivative of fuzzy-valued maps which by the intermediate of its α -cuts is defined by the so called Hukuhara derivative of multivalued maps.

In [7] the author considers a differential equation (nonlinear) in R^1 with some parameters which do not depend on time. Its general solution depending on initial condition, time, and parameters is used to define the

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“fuzzy” solution as an extension of this function through Zadeh’s extension principle.

We adopt an approach different from those mentioned above to treat ordinary differential equations in \mathbb{R}^n containing parameters which are known only through a time changing fuzzy description. The initial condition is also fuzzy. This method does not require the notion of derivative of fuzzy-valued maps and makes use of techniques of differential inclusions, the theory of which is quite well developed now. Differential inclusions have also been used to define fuzzy solutions in [4, 15], and were further developed in [11, 13]. We drop an assumption of convexity which was present in all cited papers as well as some regularity. This is discussed in detail in Section 4.

One of natural applications of this theory is the estimation of possible behavior of differential systems with uncertain parameters which are described in a fuzzy way. These applications as well as exact relations to the results in the papers cited above will be contained in a paper in preparation by the authors.

In Section 2 we give some basic notions and facts concerning multivalued maps and differential inclusions. They are of fundamental importance for this version of the definition of “fuzzy” solutions of ordinary differential equations. In Section 4 we give a definition of such a solution of a differential equation with a fuzzy initial condition and fuzzy parameters depending on time (measurable). The construction is based on the description of fuzzy sets through so called α -cuts. For each $\alpha \in (0, 1]$ we define a differential inclusion whose solutions form the α -cut for the solution that we are defining. Due to such procedure we get solutions which are fuzzy sets in the space of absolutely continuous functions. This permits us also to get solutions as “fuzzy processes.”

If one has in mind applications and in particular numerical treatment of problems, then continuous dependence on initial conditions and parameters is very important. We introduce in an arbitrary space of fuzzy sets a metric which seems particularly well adapted to this goal—called here the hypograph metric. It provides complete and separable or even compact (under some assumptions) spaces of fuzzy sets. It can be used (*mutatis mutandis*) in the space of initial conditions, parameters, and solutions, and permits one to have the required continuous dependence—Section 5. The hypograph metric seems also to agree well with the philosophy of fuzzy sets. It is near to the sendograph metric defined in [19] and used later in [12, 20]. There are, however, some important differences—the details are given in Section 3.1.

In [1, 9] attempts were made to define fuzzy solutions of differential inclusions. In [9] the actual grade of membership function over the set of solutions of a differential inclusion was built. This was done, however, in a different spirit than the variant treated in this paper.

2. PRELIMINARIES

Let (U, d) be a metric space. By $\text{Cl}(U)$ and $\text{Comp}(U)$ we shall denote respectively the families of closed and compact nonempty subsets of U .

We place a bar over a symbol if it represents a fuzzy set. Let \bar{A} be a fuzzy set in U . For $\alpha \in (0, 1]$, the α -level set of \bar{A} is

$$[\bar{A}]^\alpha = \{x \in U : \mu(x|\bar{A}) \geq \alpha\},$$

where $\mu(\cdot|\bar{A})$ is the membership function of \bar{A} . We separately define the 0-level set

$$[\bar{A}]^0 = \text{supp } \bar{A} = \text{cl}(\{x \in U : \mu(x|\bar{A}) > 0\}).$$

Among fuzzy sets in U we distinguish the class $\mathcal{FR}(U)$ of regular fuzzy sets. $\bar{A} \in \mathcal{FR}(U)$ if $[\bar{A}]^1 \neq \emptyset$, $[\bar{A}]^0 \in \text{Comp}(U)$ and if $\mu(\cdot|\bar{A})$ is upper semi-continuous.

We shall use the following representation theorem (see [23]).

THEOREM 2.1. *For every fuzzy set $\bar{A} \in \mathcal{FR}(U)$:*

- (1) $(\forall \alpha \in [0, 1]) [\bar{A}]^\alpha \in \text{Comp}(U)$,
- (2) if $0 \leq \alpha \leq \beta \leq 1$, then $[\bar{A}]^\beta \subset [\bar{A}]^\alpha$,
- (3) if $\alpha_k \in (0, 1]$ and $\alpha_k \nearrow \alpha$, then $[\bar{A}]^\alpha = \bigcap_{k \geq 1} [\bar{A}]^{\alpha_k}$.

Conversely, if a family $\{K^\alpha : 0 \leq \alpha \leq 1\}$, where $K^0 = \text{cl}(\bigcup \{K^\alpha : \alpha \in (0, 1]\})$, satisfies the above three properties (with $[\bar{A}]^\alpha$ replaced by K^α), then there exists $\bar{A} \in \mathcal{FR}(U)$ such that $[\bar{A}]^\alpha = K^\alpha$ for $0 < \alpha \leq 1$. Moreover $\mu(u|\bar{A}) = \sup\{\alpha : u \in K^\alpha\}$ for $u \in K^0$ and equals 0 outside K^0 .

For $A, B \subset U$ we put $e(A, B) = \sup\{\text{dist}(a, B) : a \in A\}$, where $\text{dist}(a, B) = \inf\{d(a, b) : b \in B\}$. The Hausdorff distance of two sets $A, B \subset U$ is defined as

$$H(A, B) = \max\{e(A, B), e(B, A)\}.$$

H is a metric in $\text{Cl}(U)$. The family $\text{Comp}(U)$ is closed in $\text{Cl}(U)$ with respect to H . We refer to [8] for the discussion of this metric space. One of important properties is that if U is compact, then $(\text{Comp}(U), H)$ also is.

We recall now the definition of measurable set-valued maps. We shall use only maps defined on some interval $I = [t_0, T]$ equipped with the Lebesgue σ -field \mathcal{L} and Lebesgue measure. The definition and the properties are valid for σ -finite, complete measure spaces.

DEFINITION 2.1. A map $F: I \rightarrow \text{Cl}(U)$ is called measurable if for every open set $\mathcal{O} \subset U$

$$\{t \in I : F(t) \cap \mathcal{O} \neq \emptyset\} \in \mathcal{L}.$$

One of the most important theorems on measurable set-valued maps is the following selection theorem due to Kuratowski and Ryll-Nardzewski.

THEOREM 2.2. [3, Theorem 8.1.3]. *If U is complete and separable, then every measurable map $F: I \rightarrow \text{Cl}(U)$ has a measurable selection φ , i.e., a measurable function $\varphi: I \rightarrow U$ such that $\varphi(t) \in F(t)$ for all $t \in I$.*

An important characterization of measurable set-valued maps uses its graph

$$\text{Gr}(F) = \{(t, u) \in I \times U : u \in F(t)\}.$$

By $\mathcal{L} \otimes \mathcal{B}(U)$ we mean the smallest σ -field in $I \times U$ containing all the sets $\Lambda \times \Gamma$ for $\Lambda \in \mathcal{L}$, $\Gamma \in \mathcal{B}(U)$ where $\mathcal{B}(U)$ is the Borel σ -field in U .

THEOREM 2.3. *A map $F: I \rightarrow \text{Cl}(U)$ is measurable iff $\text{Gr}(F) \in \mathcal{L} \otimes \mathcal{B}(U)$.*

This is a consequence of [8, Theorems III.13 and III.23].

Another useful selection theorem (which we shall need later) is the so called Filippov lemma that we shall cite now.

Let U, V be complete, separable metric spaces. A function $f: I \times U \rightarrow V$ is called a Carathéodory map when it is measurable with respect to the first variable (when the second is fixed) and continuous with respect to the second one.

THEOREM 2.4 [3, Theorem 8.2.10]. *If $f: I \times U \rightarrow V$ is a Carathéodory map, $G: I \rightarrow \text{Cl}(U)$ is measurable, $\zeta: I \rightarrow V$ is measurable, and $\zeta(t) \in f(t, G(t))$ a.e. in I , then there is a measurable selection $g(t) \in G(t)$ such that $\zeta(t) = f(t, g(t))$ a.e. in I .*

A fundamental role in our definition of fuzzy solutions of differential equations will be played by differential inclusions—there is now a rich theory that is presented in [2, 3, 8, 10, 18].

We shall cite here some basic notions and theorems that will be useful in the sequel.

Let $F: I \times \mathbb{R}^n \supset W \rightarrow \text{Cl}(\mathbb{R}^n)$. Differential inclusion is a relation

$$\dot{x} \in F(t, x). \quad (1)$$

A function $x: I \rightarrow \mathbb{R}^n$ is a Carathéodory solution of (1) if it is absolutely continuous, $(t, x(t)) \in W$ for all $t \in I$, and $\dot{x}(t) \in F(t, x(t))$ a.e. in I .

One can consider an initial value problem composed of (1) and an initial condition of the form $x(t_0) = x_0$ or $x(t_0) \in X_0 \subset \mathbb{R}^n$.

There are many theorems which give the existence of solution of (1) under different assumptions as well as theorems describing the properties of the set of solutions. Particularly fruitful is the one due to Filippov. We shall formulate it now.

Let $F: I \times \mathbb{R}^n \rightarrow \text{Cl}(\mathbb{R}^n)$ satisfy the following conditions:

- (i) $(\forall x \in \mathbb{R}^n) F(\cdot, x)$ is measurable;
- (ii) there is an integrable function $L: I \rightarrow \mathbb{R}_+$ such that for almost all $t \in I$ and all $x, y \in \mathbb{R}^n$

$$F(t, x) \subset F(t, y) + L(t) \cdot \|x - y\| \cdot B_1,$$

where B_1 is the closed unit ball in \mathbb{R}^n —this is a Lipschitz condition with respect to the Hausdorff metric;

- (iii) the map $t \rightarrow \text{dist}(0, F(t, 0))$ is integrable.

THEOREM 2.5 [3, Theorem 10.4.1]. *Let the conditions (i), (ii), (iii) hold, let $y: I \rightarrow \mathbb{R}^n$ be any absolutely continuous function, and put $\gamma(t) = \text{dist}(\dot{y}(t), F(t, y(t)))$. Then for any $x_0 \in \mathbb{R}^n$ there is a solution $x(\cdot)$ of (1) with $x(t_0) = x_0$ such that*

$$\begin{aligned} \|x(t) - y(t)\| &\leq \eta(t) \quad \text{on } I \quad \text{and} \\ \|\dot{x}(t) - \dot{y}(t)\| &\leq L(t)\eta(t) + \gamma(t) \quad \text{a.e. in } I, \end{aligned}$$

where $\eta(t) = \exp(\int_{t_0}^t L(s) ds) \cdot (\|x_0 - y(t_0)\| + \int_{t_0}^t \gamma(s) ds)$.

This theorem permits one to prove continuous dependence on the initial condition x_0 of sets of solutions of (1) and also the following important result called the Filippov–Ważewski relaxation theorem. Below, by $\overline{\text{co}}Z$ we mean the closed, convex hull of Z .

THEOREM 2.6 [3, Theorem 10.4.3]. *We assume that F satisfies assumptions (i), (ii) above and that it is bounded in the sense*

$$\sup\{\|v\| : v \in F(t, y)\} \leq \psi_1(t)\|y\| + \psi_2(t)$$

for some integrable $\psi_1(\cdot), \psi_2(\cdot)$.

Let $y(\cdot)$ be a solution to the ‘relaxed’ inclusion

$$\dot{y} \in \overline{\text{co}} F(t, y). \quad (2)$$

Then for every $\varepsilon > 0$ there is a solution $x(\cdot)$ to (1) such that $x(t_0) = y(t_0)$ and

$$\max\{\|x(t) - y(t)\| : t \in I\} \leq \varepsilon.$$

COROLLARY 2.1. *If F satisfies (i), (ii), (iii) and is integrably bounded, then the set of solutions to (1) is dense in the set of solutions to (2) in the topology of uniform convergence.*

3. HYPOGRAPH METRIC AND MEASURABLE FUZZY-VALUED MAPS

3.1. Hypograph Metric

Let (U, d) be a complete metric space. By a hypograph of $\bar{A} \in \mathcal{FR}(U)$ we mean the set

$$h(\bar{A}) = \{(u, r) \in U \times [0, 1]; r \leq \mu(u \mid \bar{A})\}$$

We introduce in $\mathcal{FR}(U)$ the metric

$$\Delta_U(\bar{A}, \bar{B}) = H'(h(\bar{A}), h(\bar{B})),$$

where H' denotes the Hausdorff metric generated in the family $\text{Cl}(U \times [0, 1])$ of closed subsets of $U \times [0, 1]$ by the metric

$$d'((u_1, r_1), (u_2, r_2)) = \max\{d(u_1, u_2), |r_1 - r_2|\}.$$

The following proposition is easily proved.

PROPOSITION 3.1. *The family of hypographs $\{h(\bar{A}); \bar{A} \in \mathcal{FR}(U)\}$ is closed in $\text{Cl}(U \times [0, 1])$ in the topology of H' and so the metric space $(\mathcal{FR}(U), \Delta_U)$ is complete. If (U, d) is compact, then $(\mathcal{FR}(U), \Delta_U)$ also is. (See [8, Theorem II-3 and the remark after Theorem II-4].)*

The family of fuzzy sets with quasi-concave membership functions and the family of fuzzy sets with membership functions concave over their supports are closed in the metric space $(\mathcal{FR}(U), \Delta_U)$.

The metric Δ_U has been studied in [5, 6], where many of its important properties can be found. Here is a proposition which gives a characterization of convergence in Δ_U [6, Lemma 1.5.].

PROPOSITION 3.2. *If U is compact, then a sequence \bar{A}_m converges to \bar{A}_0 in Δ_U if and only if the following two conditions are true:*

- (a) *If $u_m \rightarrow u$, then $\limsup_{m \rightarrow \infty} \mu(u_m \mid \bar{A}_m) \leq \mu(u \mid \bar{A}_0)$*
- (b) *There is a sequence $u_m \rightarrow u$ for which $\liminf_{m \rightarrow \infty} \mu(u_m \mid \bar{A}_m) \geq \mu(u \mid \bar{A}_0)$.*

Let us compare Δ_U to the metric D introduced in [24] and used in [16, 17]. It is defined by

$$D(\bar{A}, \bar{B}) = \sup_{\alpha \in (0, 1]} H([\bar{A}]^\alpha, [\bar{B}]^\alpha),$$

where H denotes the Hausdorff metric in U .

LEMMA 3.1. *Γ is essentially stronger than Δ_U .*

Proof. Take $\bar{A}, \bar{B} \in \mathcal{FR}(U)$ and fix any $(u, r) \in h(\bar{A})$. We have

$$\text{dist}((u, r), h(\bar{B})) \leq \text{dist}(u, [\bar{B}]^r) \leq \Gamma(\bar{A}, \bar{B})$$

and thus

$$\sup\{\text{dist}((u, r), h(\bar{B})); (u, r) \in h(\bar{A})\} \leq \Gamma(\bar{A}, \bar{B}).$$

This inequality and the one where the roles of \bar{A} and \bar{B} are interchanged imply

$$\Delta_U(\bar{A}, \bar{B}) \leq \Gamma(\bar{A}, \bar{B}) \quad \text{for all } \bar{A}, \bar{B} \in \mathcal{FR}(U)$$

In order to see that Γ is not equivalent to Δ_U consider $U = [0, 1]$ and the sequence \bar{A}_m defined by

$$\mu(u | \bar{A}_m) = \frac{1}{m}u + 1 - \frac{1}{m}.$$

For \bar{A}_0 defined by $\mu(u | \bar{A}_0) \equiv 1$ we have $\Delta_U(\bar{A}_m, \bar{A}_0) \rightarrow 0$ whereas $\Gamma(\bar{A}_m, \bar{A}_0) = 1$ for all $m \in N$.

For U compact the space $(\mathcal{FR}(U), \Delta_U)$ is compact and $(\mathcal{FR}(U), D)$ is not even separable.

It can be easily seen that if a sequence $\mu(\cdot, \bar{A}_m)$ converges uniformly to some function, then this limit function is a grade of membership function $\mu(\cdot, \bar{A}_0)$ of some $\bar{A}_0 \in \mathcal{FR}(U)$ and $\Delta_U(\bar{A}_m, \bar{A}_0) \rightarrow 0$. The inverse is not true—the convergence $\Delta_U(\bar{A}_m, \bar{A}_0) \rightarrow 0$ does not imply the uniform convergence of $\mu(\cdot | \bar{A}_m)$ to $\mu(\cdot | \bar{A}_0)$.

Apart from good properties as a metric, Δ_U seems to be natural if we think of interpretations of fuzzy sets.

The hypograph metric is very near to the sendograph metric introduced by Kloeden in [19]. By the sendograph of $\bar{A} \in \mathcal{FR}(U)$ we mean the set

$$\text{send}(\bar{A}) = h(\bar{A}) \cap ([\bar{A}]^0 \times [0, 1])$$

and the sendograph metric is defined as

$$\mathcal{S}(\bar{A}, \bar{B}) = H'(\text{send}(\bar{A}), \text{send}(\bar{B})).$$

It is clear that $\Delta_U(\bar{A}, \bar{B}) \leq \mathcal{S}(\bar{A}, \bar{B})$ and also that there exist sequences convergent with respect to Δ_U and not convergent with respect to \mathcal{S} . The difference is that the convergence $\mathcal{S}(\bar{A}_k, \bar{A}_0) \rightarrow 0$ requires the convergence of supports $H([\bar{A}_k]^0, [\bar{A}_0]^0) \rightarrow 0$ contrary to the case $\Delta_U(\bar{A}_k, \bar{A}_0) \rightarrow 0$. Whether one wishes to have the supports convergent or not is a question of the interpretation of fuzziness, but from the mathematical point of view the advantage of Δ_U is Proposition 3.1, which is not true for \mathcal{S} .

3.2. Measurable Fuzzy-Valued Maps

Let (U, d) be complete and separable. With every map $\bar{\Phi}: I \rightarrow \mathcal{FR}(U)$ we can associate the multivalued map which assigns the hypograph $h(\bar{\Phi}(t))$ to each $t \in I$. We shall denote it as $h(\bar{\Phi}(\cdot))$.

DEFINITION 3.1. We call a map $\bar{\Phi}: I \rightarrow \mathcal{FR}(U)$ measurable if the corresponding multivalued map $h(\bar{\Phi}(\cdot))$ is measurable in the sense of Definition 2.1.

This condition is equivalent to the measurability of $\bar{\Phi}$ considered as a map from I to $\mathcal{FR}(U)$ endowed with the Hausdorff metric H' (see [8, Chapt. III]).

We shall show that the above property of measurability can be characterized by the measurability of α -cuts of $\bar{\Phi}$.

PROPOSITION 3.3. A map $\bar{\Phi}: I \rightarrow \mathcal{FR}(U)$ is measurable if and only if for each $\alpha \in (0, 1]$ the multivalued map $[\bar{\Phi}(\cdot)]^\alpha$ is measurable.

Proof. Suppose first that $\bar{\Phi}(\cdot)$ is measurable. That means that the multivalued map $h(\bar{\Phi}(\cdot))$ is measurable and thus its graph belongs to the σ -field $\mathcal{L} \otimes \mathcal{B}(U \times [0, 1])$. For any fixed $\alpha \in (0, 1]$ the set

$$\begin{aligned} Z_\alpha &= \{(t, u, \alpha) \in I \times U \times [0, 1]; u \in [\bar{\Phi}(t)]^\alpha\} \\ &= \text{Gr}(h(\bar{\Phi}(\cdot))) \cap (I \times U \times \{\alpha\}) \end{aligned}$$

belongs to $\mathcal{L} \otimes \mathcal{B}(U \times [0, 1])$. This implies that

$$\text{Gr}([\bar{\Phi}(\cdot)]^\alpha) = \{(t, u) \in I \times U; (t, u, \alpha) \in Z_\alpha\} \in \mathcal{L} \otimes \mathcal{B}(U),$$

which means that $[\bar{\Phi}(\cdot)]^\alpha$ is measurable.

Inversely, suppose that all the multivalued maps $[\bar{\Phi}(\cdot)]^\alpha$, for $\alpha \in (0, 1]$, are measurable and so their graphs belong to $\mathcal{L} \otimes \mathcal{B}(U)$. (For $\alpha = 0$ this graph equals $I \times U$.)

Fix an arbitrary denumerable set $\{\alpha_m; m \in N\}$ dense in $[0, 1]$ and containing 0. For each m the multivalued map

$$t \rightarrow [\bar{\Phi}(t)]^{\alpha_m} \times \{\alpha_m\}$$

is measurable as its graph is equal to $\text{Gr}([\bar{\Phi}(\cdot)]^{\alpha_m}) \times \{\alpha_m\}$ and belongs to $\mathcal{L} \otimes \mathcal{B}(U \times [0, 1])$. In view of [3, Theorem 8.2.4] the multivalued map

$$t \rightarrow \text{cl}_{U \times [0, 1]} \bigcup_{m \in N} ([\bar{\Phi}(t)]^{\alpha_m} \times \{\alpha_m\})$$

is measurable— $\text{cl}_{U \times [0, 1]}$ denotes the closure in $U \times [0, 1]$.

Now it is enough to observe that due to the upper semicontinuity of $\mu(\cdot|\bar{\Phi}(t))$ —which is equivalent to the closedness of the hypographs $h(\bar{\Phi}(t))$ —we have for every $t \in I$ the equality

$$\text{cl}_{U \times [0,1]} \bigcup_{m \in N} ([\bar{\Phi}(t)]^{\alpha_m} \times \{\alpha_m\}) = h(\bar{\Phi}(t)),$$

and this ends the proof.

The above proposition shows that in the case of convex fuzzy sets our definition of measurability is equivalent to strong measurability in [12, Chapt. 10].

We shall denote by $\mathcal{M}(I, \mathcal{FR}(U))$ the family of equivalence classes of almost everywhere equal measurable maps $\bar{\Phi}: I \rightarrow \mathcal{FR}(U)$. We shall, as usually, work with representatives of equivalence classes. This space will be equipped with the metric

$$D(\bar{\Phi}, \bar{\Psi}) = \text{ess sup} \{ \Delta_U(\bar{\Phi}(t), \bar{\Psi}(t)) \}, \quad (3)$$

where ess sup stands for the essential supremum of a function, that is, the least number c such that the set of arguments for which the value of the function is greater or equal c is of measure zero.

4. DEFINITION OF FUZZY SOLUTIONS OF DIFFERENTIAL EQUATIONS WITH FUZZY INITIAL CONDITION AND PARAMETERS

Let $(U_1, d_1), \dots, (U, d_p)$ be separable, complete metric spaces and consider a map

$$f: I \times \mathbb{R}^n \times U_1 \times \dots \times U_p \rightarrow \mathbb{R}^n.$$

We think of a situation when the change in time of some model is described by a system of ordinary differential equations in \mathbb{R}^n

$$\dot{x} = f(t, x, \varphi_1(t), \dots, \varphi_p(t)), \quad x(t_0) = x_0 \quad (4)$$

containing changing in time parameters $\varphi_i(t)$ which are not known exactly but only “fuzzy” information is available. The initial condition is also described in a fuzzy way. It means that instead of $\varphi_i: I \rightarrow U_i$ we shall use the functions $\bar{\Phi}_i \in \mathcal{M}(I, \mathcal{FR}(U_i))$ and instead of $x_0 \in \mathbb{R}^n$ we use a fuzzy set $\bar{X}_0 \in \mathcal{FR}(\mathbb{R}^n)$.

Our aim is to describe the dynamics of such systems in a fuzzy way. We shall do it by defining a solution which will be a fuzzy set in the space of absolutely continuous functions. It will yield also a fuzzy solution in the sense of fuzzy processes.

The function f will satisfy the following conditions:

(i) *there is an integrable function $L: I \rightarrow R_+$ such that a.e. in I for all $x, y \in \mathbb{R}^n$ and $u_i \in U_i$*

$$\|f(t, x, u_1, \dots, u_p) - f(t, y, u_1, \dots, u_p)\| \leq L(t)\|x - y\|;$$

(ii) *for almost all $t \in I$ the map $(u_1, \dots, u_p) \rightarrow f(t, x, u_1, \dots, u_p)$ is continuous for every fixed $x \in \mathbb{R}^n$;*

(iii) *for every $(x, u_1, \dots, u_p) \in \mathbb{R}^n \times U_1 \times \dots \times U_p$ the map $t \rightarrow f(t, x, u_1, \dots, u_p)$ is measurable;*

(iv) *there is an integrable function $\psi: I \rightarrow R_+$ such that for all t, u_1, \dots, u_p*

$$\|f(t, 0, u_1, \dots, u_p)\| \leq \psi(t).$$

Let us remark that conditions (i) and (iv) imply that for all t, x, u_1, \dots, u_p the following inequality holds

$$\|f(t, x, u_1, \dots, u_p)\| \leq L(t)\|x\| + \psi(t). \quad (5)$$

In our considerations we shall use the following one-parameter family of multivalued maps:

$$F^\alpha(t, x) = f(t, x, [\bar{\Phi}_1(t)]^\alpha, \dots, [\bar{\Phi}_p(t)]^\alpha). \quad (6)$$

The assumptions on f , measurability of $\bar{\Phi}_i(\cdot)$, and Proposition 3.3 imply that every map

$$F^\alpha: I \times \mathbb{R}^n \rightarrow \text{Comp}(\mathbb{R}^n)$$

satisfies the following properties:

(I) *for almost all $t \in I$*

$$\forall x, y \in \mathbb{R}^n \quad F^\alpha(t, x) \subset F^\alpha(t, y) + L(t)\|x - y\|B_1 \quad (7)$$

where B_1 is the closed unit ball in \mathbb{R}^n ;

(II) *the maps $t \rightarrow F^\alpha(t, x)$ are measurable;*

(III) *$\sup\{\|v\|: v \in F^\alpha(t, x)\} \leq L(t)\|x\| + \psi(t)$ for almost all $t \in I$.*

The tool which will serve us to define a solution of (4) with fuzzy initial condition and fuzzy parameters will be the following one-parameter family of differential inclusions with multivalued initial conditions:

$$\dot{x} \in F^\alpha(t, x), \quad x(t_0) \in [\bar{X}_0]^\alpha. \quad (8)$$

Due to Theorem 2.5 for every α and every initial point in $[\bar{X}_0]^\alpha$ the set of Carathéodory solutions of (8) on I is nonempty. We shall denote by W^α its closure in the topology of uniform convergence.

The Filippov–Ważewski theorem (see Corollary 2.1) implies that W^α is for each α the set of Carathéodory solutions of the following differential inclusion:

$$\dot{x} \in \overline{\text{co}} F^\alpha(t, x), \quad x(t_0) \in [\bar{X}_0]^\alpha. \quad (9)$$

We gather the properties of the family of sets of functions W^α in the following theorem:

THEOREM 4.1. *Under the assumptions (i), (ii), (iii), (iv) and for measurable fuzzy-valued functions $\bar{\Phi}_1(\cdot), \dots, \bar{\Phi}_p(\cdot)$ the following properties hold:*

1. $W^1 \neq \emptyset$;
2. W^α are compact and connected;
3. if $0 < \alpha_1 \leq \alpha_2 \leq 1$ then $W^{\alpha_1} \supset W^{\alpha_2}$;
4. if $\alpha_k \in (0, 1]$ and $\alpha_k \nearrow \alpha_0$ then $W^{\alpha_0} = \bigcap_{k \geq 1} W^{\alpha_k}$.

Proof. (1) It is a consequence of non-emptiness of sets $F^1(t, x)$ and Theorem 2.5.

(2) The compactness is proved in [2, Theorem 2.2.1] and the connectedness is there given as a remark after the proof of Corollary 2.2.5.

(3) This is implied by the inclusions $F^{\alpha_1}(t, x) \supset F^{\alpha_2}(t, x)$ true a.e. in I for all $x \in \mathbb{R}^n$.

(4) The function f is continuous with respect to (u_1, \dots, u_p) ; the sets $[\bar{\Phi}_i(t)]^\alpha$ are nonempty, compact, and decreasing with respect to α . Moreover

$$\bigcap_{k \geq 1} [\bar{\Phi}_i(t)]^{\alpha_k} = [\bar{\Phi}_i(t)]^{\alpha_0};$$

thus

$$\bigcap_{k \geq 1} F^{\alpha_k}(t, x) = F^{\alpha_0}(t, x).$$

The sets $F^\alpha(t, x)$ are nonempty, compact, and decreasing with respect to α and so the last equality yields

$$\bigcap_{k \geq 1} \overline{\text{co}} F^{\alpha_k}(t, x) = \overline{\text{co}} F^{\alpha_0}(t, x).$$

Moreover we have $\bigcap_{k \geq 1} [\bar{X}_0]^{\alpha_k} = [\bar{X}_0]^{\alpha_0}$. Altogether this implies the desired equality by standard arguments.

Properties 1, 2, 3, and 4 in Theorem 4.1 suggest that we can treat the sets W^α as α -cuts of some fuzzy set in the space $AC(I, \mathbb{R}^n)$ of absolutely continuous functions defined on I with values in \mathbb{R}^n .

DEFINITION 4.1. By a fuzzy solution of differential equation (4) with fuzzy initial condition \bar{X}_0 and fuzzy parameters $\bar{\Phi}(t), \dots, \bar{\Phi}(t)$ depending on $t \in I$ we shall mean the fuzzy subset of $AC(I, \mathbb{R}^n)$ whose α -cuts are the sets W^α . We shall denote this solution by $\bar{\Sigma}(\bar{X}_0, \bar{\Phi}_1, \dots, \bar{\Phi}_p)$. (We can thus write $[\bar{\Sigma}(\bar{X}_0, \bar{\Phi}_1, \dots, \bar{\Phi}_p)]^\alpha = W^\alpha$.)

Theorem 4.1 guarantees that under our assumptions on f , \bar{X}_0 , and fuzzy parameters, the fuzzy solution defined in Definition 4.1 exists. It is also obviously unique.

Using Definition 4.1 which says what we mean by a fuzzy solution in the space $AC(I, \mathbb{R}^n)$ we can define a “fuzzy process solution.”

DEFINITION 4.2. By a fuzzy process solution of (4) on the interval I with fuzzy initial condition \bar{X}_0 and fuzzy parameters $\bar{\Phi}_1, \dots, \bar{\Phi}_p$ we shall mean the fuzzy process $\bar{X}: I \rightarrow \mathbb{R}^n$ defined for every $t \in I$ by the family of α -cuts

$$[\bar{X}(t)]^\alpha = \{x(t) : x(\cdot) \in [\bar{\Sigma}_I(\bar{X}_0, \bar{\Phi}_1, \dots, \bar{\Phi}_p)]^\alpha\}.$$

Knowing a fuzzy solution we can find the corresponding fuzzy process solution but not inversely. However, the fuzzy process solution is usually much easier to find and to handle with than the fuzzy solution is. It is possible to build approximate methods to compute the fuzzy process solutions while doing the same for the fuzzy solutions is much harder. The difference is similar to finding attained sets for differential inclusions and the sets of all solutions—it is not necessary to know the solutions in order to find the attained sets. There exist numerical methods for computing attained sets; see for example [22].

DEFINITION 4.3. We shall say that functions

$$x(\cdot) \in [\bar{\Sigma}(\bar{X}_0, \bar{\Phi}_1, \dots, \bar{\Phi}_p)]^\alpha$$

are realizations on the level α (or, shortly, α -realizations) of the corresponding fuzzy process solution $\bar{X}(\cdot)$.

The remaining part of this section is devoted to some comments on relations to results existing in the literature. First of all we would like to underline that we do not speak in this paper about fuzzy differential equations—we did not try to define the meaning of such a notion. We tried to define what a solution of a usual ordinary differential equation could mean when the knowledge of some parameters and initial conditions is not exact but fuzzy. It is evident that this solution should also be fuzzy.

It seems that the approach based on differential inclusions is quite natural and well adapted to the "fuzzy" situation. It is so natural that it has been invented and reinvented independently at least three times. After the first version of this text had been prepared we found the paper by Baidosov [4] who probably was the first one to consider this idea. At the beginning Baidosov defines a general notion of solution of something that he calls fuzzy differential inclusion

$$\dot{x} \in \bar{\Psi}(t, x), \quad (10)$$

where $\bar{\Psi}: G \supset I \times \mathbb{R}^n \rightarrow \mathcal{FR}(\mathbb{R}^n)$ is a fuzzy valued function of crisp variables. He understands Eq. (10) as a symbol and does not use anything like $\dot{x}(t) \in \bar{\Psi}(t, x(t))$ or $\dot{\bar{x}}(t) \in \bar{\Psi}(t, \bar{x}(t))$ with fuzzy sets at the right-hand side. He directly defines a solution as a fuzzy set $\bar{R}[I]$ in $AC(I, \mathbb{R}^n)$ putting

$$\mu(x(\cdot) \mid \bar{R}[I]) = \text{ess inf } \mu(\dot{x}(t) \mid \bar{\Psi}(t, x(t))).$$

Here ess inf stands for the essential infimum of a function, i.e., the greatest lower bound valid almost everywhere. It is evident that $\mu(x(\cdot) \mid \bar{R}[I]) \geq \alpha$ if and only if $\dot{x}(t) \in [\bar{\Psi}(t, x(t))]^\alpha$ a.e. in I . Baidosov applied his definition to get a notion of solution of a differential equation $\dot{x} = f(t, x, \phi(t))$ when the information about the parameters $\phi(t)$ is given in a fuzzy way $\bar{\Phi}(t)$ under the assumptions that the multivalued maps $t \rightarrow [\bar{\Phi}(t)]^\alpha$ are upper semicontinuous. f is assumed to be continuous and the sets $f(t, x, [\bar{\Phi}(t)]^\alpha)$ convex for all t, x , and α . This assumption of convexity is rather restrictive. It may happen that for some fixed t, x , and \bar{A} the sets $f(t, x, [\bar{A}]^\alpha)$ are convex but this is not true if we replace \bar{A} by its translation defined by $\mu(u \mid \bar{B}) = \mu(u + a \mid \bar{A})$. The initial conditions are not considered in [4] and so the initial value problem is not treated. In consequence in the crisp case his definition of the solution gives a membership function which is equal to 1 for every usual solution and 0 for any other function. At the end a short discussion of the case when f is measurable with respect to t is included.

The referees brought our attention to [15] where the idea of applying differential inclusions was also applied to define the solution of (10) but in addition the initial value problem was treated. The assumptions are practically the same as in [4] in particular the convexity of $[\bar{\Psi}(t, x)]^\alpha$. The definition was done directly through reachable sets and the solution as a fuzzy subset of $AC(I, \mathbb{R}^n)$ was not considered. The author gave also a justification of this method of defining fuzzy solutions.

In [11, 13] the authors continued the idea from [15]; in particular the regularity of the right-hand side was weakened. In [15] continuity in t and the Lipschitz condition with respect to x were required and in [11] upper semicontinuity with respect to the joint variable (t, x) was sufficient.

Let us discuss still shortly the relation to the ideas on fuzzy dynamical systems described in [20]. We consider the case when f does not depend on t and the fuzzy parameters are also constant. Equation (4) then becomes

$$\dot{x} = f(x, \phi_1, \dots, \phi_p).$$

Assume that the fuzzy sets $\bar{\Phi}_1, \dots, \bar{\Phi}_p$ which replace ϕ_1, \dots, ϕ_p are also constant. Let $\mathcal{A}^\alpha(t, y)$ denote the set of points which can be reached at time t by the solutions of

$$\dot{x} \in \overline{\text{co}} F^\alpha(x), \quad x(0) = y.$$

The Lipschitz property of F^α implies that for each α the map $(t, y) \rightarrow \mathcal{A}^\alpha(t, y)$ is continuous and so these maps are generalized semi-dynamical systems from [20, Sect. 3]. Simple examples show, however, that fuzzy valued map $(t, y) \rightarrow \bar{\mathcal{A}}(t, y)$, where $[\bar{\mathcal{A}}(t, y)]^\alpha = \mathcal{A}^\alpha(t, y)$, does not define a fuzzy dynamical system defined in [20, Sect. 4].

5. CONTINUOUS DEPENDENCE ON THE INITIAL CONDITION AND PARAMETERS

We shall assume that some compact set $K_0 \subset \mathbb{R}^n$ contains the supports of all fuzzy initial conditions that we shall consider and also that the spaces $(U_1, d_1), \dots, (U, d_p)$ are compact. We introduce in

$$\Lambda = \mathcal{FR}(K_0) \times \mathcal{M}(I, \mathcal{FR}(U_1)) \times \dots \times \mathcal{M}(I, \mathcal{FR}(U_p))$$

the metric

$$\begin{aligned} \omega((\bar{X}_0, \bar{\Phi}_1, \dots, \bar{\Phi}_p), (\bar{Y}_0, \bar{\Psi}_1, \dots, \bar{\Psi}_p)) \\ = \max\{\Delta_{K_0}(\bar{X}_0, \bar{Y}_0), D_1(\bar{\Phi}_1, \bar{\Psi}_1), \dots, D_p(\bar{\Phi}_p, \bar{\Psi}_p)\}, \end{aligned}$$

where D_i denotes the metrics in $\mathcal{M}(I, \mathcal{FR}(U_i))$ defined by (3).

Using this metric we can now formulate the theorem on continuous dependence of fuzzy solutions of (4) on fuzzy initial condition and fuzzy parameters. We shall write AC instead of the whole $\mathcal{AC}(I, \mathbb{R}^n)$.

THEOREM 5.1. *The map*

$$(\bar{X}_0, \bar{\Phi}_1, \dots, \bar{\Phi}_p) \rightarrow \bar{\Sigma}(\bar{X}_0, \bar{\Phi}_1, \dots, \bar{\Phi}_p)$$

is uniformly continuous on the metric space (Λ, ω) into the space $(\mathcal{FR}(\text{AC}), \Delta_{\text{AC}})$.

Proof. Let $b_0 = \max\{\|y\|; y \in K_0\}$ and

$$b = \exp\left(\int_{t_0}^T L(u) du\right) b_0 + \int_{t_0}^T \exp\left(\int_s^T L(u) du\right) \psi(s) ds.$$

Put $K = \{x \in \mathbb{R}^n; \text{dist}(x, K_0) \leq b\}$. This is the set which in view of property III in Section 4 contains all possible α -cuts $[\bar{X}(t)]^\alpha$ of our fuzzy process solutions.

For $\delta > 0$, $t \in I$ let

$$h_\delta(t) = \max\{\|f(t, x, \varphi_1, \dots, \varphi_p) - f(t, x, \psi_1, \dots, \psi_p)\|; x \in K, d_i(\varphi_i, \psi_i) \leq \delta\}.$$

For each fixed δ the map $h_\delta(\cdot)$ is measurable and due to the uniform continuity of the function $(x, \varphi_1, \dots, \varphi_p) \rightarrow f(x, \varphi_1, \dots, \varphi_p)$ on $K \times U_1 \times \dots \times U_p$ for fixed $t \in K$ we have $\lim_{\delta \rightarrow 0+} h_\delta(t) = 0$ a.e. in I . We have also the inequality $h_\delta(t) \leq 2(L(t)b + \psi(t))$ and so the Lebesgue dominated convergence theorem implies

$$\lim_{\delta \rightarrow 0+} \int_I h_\delta(t) dt = 0. \quad (11)$$

Let us fix $\varepsilon > 0$. We show that for sufficiently small $\delta > 0$ if

$$\omega((\bar{X}_0, \bar{\Phi}_1, \dots, \bar{\Phi}_p), (\bar{Y}_0, \bar{\Psi}_1, \dots, \bar{\Psi}_p)) \leq \delta \quad (12)$$

then

$$\Delta_{AC}(\bar{\Sigma}(\bar{X}_0, \bar{\Phi}_1, \dots, \bar{\Phi}_p), \bar{\Sigma}(\bar{Y}_0, \bar{\Psi}_1, \dots, \bar{\Psi}_p)) \leq \varepsilon. \quad (13)$$

To this end it is enough to prove that if we fix an arbitrary $(\bar{X}_0, \bar{\Phi}_1, \dots, \bar{\Phi}_p) \in \Lambda$ and take a pair

$$(x(\cdot), r) \in h(\bar{\Sigma}(\bar{X}_0, \bar{\Phi}_1, \dots, \bar{\Phi}_p)) \quad (14)$$

then (12) implies the existence of a pair

$$(y(\cdot), s) \in h(\bar{\Sigma}(\bar{Y}_0, \bar{\Psi}_1, \dots, \bar{\Psi}_p))$$

such that $\max\{\max_{t \in I} \|x(t) - y(t)\|, |r - s|\} \leq \varepsilon$.

In view of (6), (9), and Definition 4.1 the inclusion (14) is equivalent to

$$\dot{x}(t) \in \overline{\text{co}} f(t, x(t), [\bar{\Phi}_1(t)]^r, \dots, [\bar{\Phi}_p(t)]^r) \quad \text{a.e. in } I, \quad x(t_0) \in [\bar{X}_0]^r$$

Due to Theorem 2.6 there is $z(\cdot) \in AC$ such that

$$\dot{z}(t) \in f(t, z(t), [\bar{\Phi}_1(t)]^r, \dots, [\bar{\Phi}_p(t)]^r) \quad \text{a.e. in } I, \quad z(t_0) = x(t_0)$$

and

$$\|x(t) - z(t)\| \leq \frac{\varepsilon}{2} \quad \text{for } t \in I. \quad (15)$$

Using Lemma 2.4 we can fix measurable selections

$$\varphi_1(t) \in [\bar{\Phi}_1(t)]^r, \dots, \varphi_p(t) \in [\bar{\Phi}(t)]^r \quad \text{for } t \in I$$

satisfying almost everywhere in I

$$\dot{z}(t) = f(t, z(t), \varphi_1(t), \dots, \varphi_p(t)).$$

Our first restriction on δ will be $\delta \leq \varepsilon$. We can fix, using (12) and the definition of the metric ω , a pair $(y_0, s) \in h(\bar{Y}_0)$ for which $\|x(t_0) - y_0\| \leq \delta$ and $|s - r| \leq \delta$. Also by (12) we know that the multivalued maps

$$t \rightarrow (\text{cl}(B_i(\varphi_i(t), \delta)) \times [r - \delta, r + \delta]) \cap h(\bar{\Psi}_i(t))$$

have a.e. in I nonempty values for $i = 1, \dots, p$. ($B_i(u, \delta)$ denotes balls in (U_i, d_i) with center at u and radius δ .) These multivalued maps are measurable (see [3, Theorem 8.2.4]) and thus admit measurable selections. We fix for each $i = 1, \dots, p$ such a selection and denote it as $(\psi_i(\cdot), \sigma_i(\cdot))$.

Put

$$\gamma = \min\{r, s, \text{essinf } \sigma_1(\cdot), \dots, \text{essinf } \sigma_p(\cdot)\},$$

where the essential infimum is taken over I . Then $\max\{0, r - \delta\} \leq \gamma \leq r$ and $0 \leq r - \gamma \leq \delta$.

We have

$$y_0 \in [\bar{Y}_0]^\gamma \quad \text{and} \quad \psi_i(t) \in [\bar{\Psi}_i(t)]^\gamma \quad \text{a.e. in } I \quad \text{for } i = 1, \dots, p. \quad (16)$$

Let now $y(\cdot)$ be a solution on I of differential equation

$$\dot{y} = f(t, y, \psi_1(t), \dots, \psi_p(t)), \quad y(t_0) = y_0. \quad (17)$$

By (16) and Definition 4.1 $y(\cdot) \in [\bar{\Sigma}(\bar{Y}_0, \bar{\Psi}_1, \dots, \bar{\Psi}_p)]^\gamma$. We shall estimate now the distance between the functions $y(\cdot)$ and $z(\cdot)$.

We have the inequality

$$\begin{aligned} & \|\dot{z}(t) - f(t, z(t), \psi_1(t), \dots, \psi_p(t))\| \\ &= \|f(t, z(t), \varphi_1(t), \dots, \varphi_p(t)) - f(t, z(t), \psi_1(t), \dots, \psi_p(t))\| \\ &\leq h_\delta(t) \end{aligned} \quad (18)$$

which in view of Theorem 2.5 implies

$$\begin{aligned} \|z(t) - y(t)\| &\leq \|z(t_0) - y(t_0)\| \exp\left(\int_{t_0}^t L(s) ds\right) \\ &\quad + \int_{t_0}^t \exp\left(\int_s^t L(\tau) d\tau\right) h_\delta(s) ds \\ &\leq M \|x(t_0) - y_0\| + M \int_I h_\delta(t) dt, \end{aligned} \quad (19)$$

where $M = \exp(\int_I L(s) ds)$. Now, if δ apart from $\delta \leq \varepsilon$ satisfies also

$$\delta \leq \frac{\varepsilon}{4M} \quad \text{and} \quad \int_I h_\delta(t) dt \leq \frac{\varepsilon}{4M},$$

we get by (19) and (15)

$$\max_{t \in I} \|x(t) - y(t)\| \leq \varepsilon. \quad (20)$$

So finally we have a pair

$$(y(\cdot), \gamma) \in h(\bar{\Sigma}(\bar{Y}_0, \bar{\Psi}_1, \dots, \bar{\Psi}_p))$$

such that (20) holds and $|\gamma - r| \leq \varepsilon$. Since δ depends only on ε for fixed f and K_0 , this ends the proof.

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